Problem K
Kitchen Knobs

Submits: 52
Accepted: at least 1

First solved by: UW1
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01:24:54

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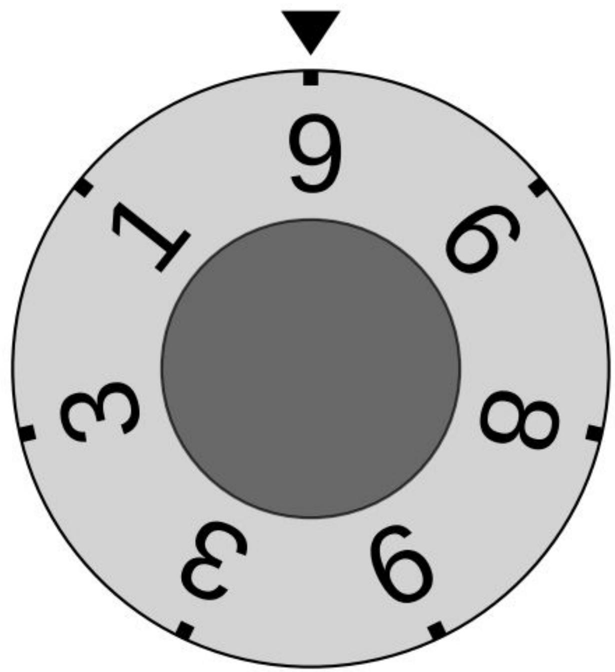
Weird kitchen knobs with 7 non-zero digits. The power of a kitchen element is the number you get from reading the digits clockwise starting from the top position.



Power: 9689331

We have a sequence of N kitchen elements, and can rotate any consecutive subsequence of kitchen knobs by an arbitrary degree in a single step.

Find the smallest number of steps to get maximum power on each element.

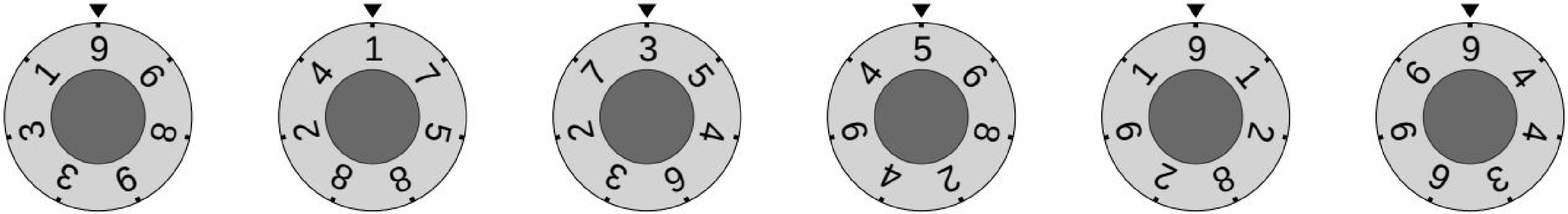
Because we have exactly 7 digits on each knob, every element either has:

a) all digits the same, in which case it's always at maximal power, or

b) exactly one position in which the maximal power is achieved.

We can pretend as if knobs of type a) didn't exist, and simplify the problem statement:

Given a sequence A with elements from [0, 6], find the smallest number of operations to make every element equal to 0. In a single operation we can add k to each number in an arbitrary subsequence of A (modulo 7).



0	3	6	5	5	5
			+ 4		
0	3	3	2	2	2
	+ 4				
0	0	0	2	2	2
				+ 5	
0	0	0	0	0	0

Let define another sequence B: B[i] = A[i] − A[i − 1]

A: 1 5 6 2 2 0 5 2 3

B: 1 4 1 3 0 5 5 4 1 4

+ 2

A: 1 5 **1** **4** **4** **2** **0** **4** 3

B: 1 4 **3** 3 0 5 5 4 **6** 4

Observe what happens to sequence B as we apply the operation to sequence A.

Once again we can simplify the problem:

Given a set B with elements from [0, 6], find the smallest number of operations to make every element equal to 0. In a single operation we can add k to any number in the set and subtract k from any other number in the set (modulo 7).

1	4	1	3	0	5	5	4	1	4
	-2					+2			
1	2	1	3	0	5	0	4	1	4
	-2				+2				
1	0	1	3	0	0	0	4	1	4
			-3				+3		
1	0	1	0	0	0	0	0	1	4
+1		-1							
2	0	0	0	0	0	0	0	1	4
+1								-1	
3	0	0	0	0	0	0	0	0	4
+4									-4
0	0	0	0	0	0	0	0	0	0

Observation: Given any set of N numbers that add up to 0 (modulo 7), we can make all numbers zero in $N - 1$ operations.

In each operation take any two non-zero numbers from the set, and make one of them zero. If there are only two numbers left, it is guaranteed they will both become zero after the last operation.

Simplifying the problem even further:

Given a set B with elements from [0, 6], group them into as many groups as possible such that the sum of each group is 0 (modulo 7).

B:	1	4	1	3	0	5	5	4	1	4

Simplifying the problem even further:

Given a set B with elements from [0, 6], group them into as many groups as possible such that the sum of each group is 0 (modulo 7).

B: 1 4 1 3 5 5 4 1 4

0

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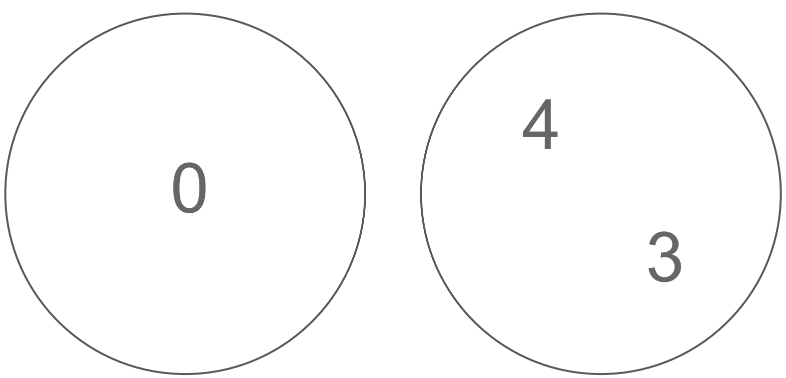
Given a set B with elements from [0, 6], group them into as many groups as possible such that the sum of each group is 0 (modulo 7).

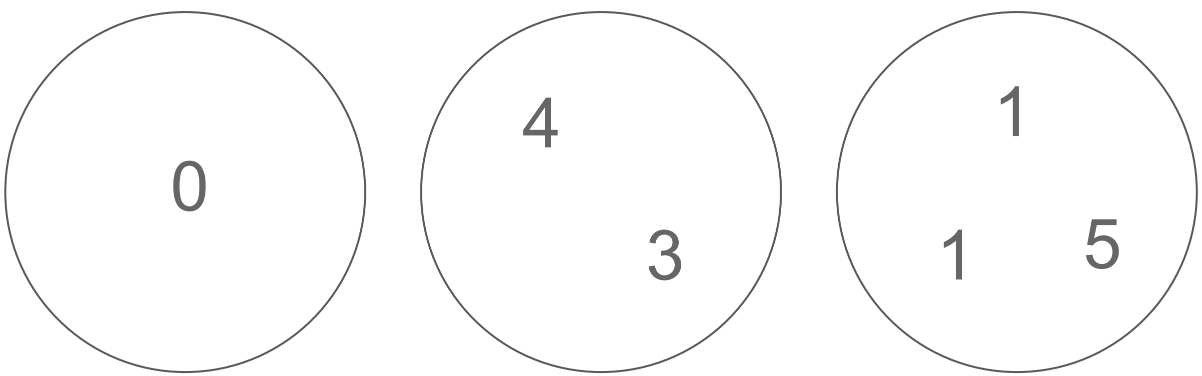
B: 1 1 5 5 4 1 4



Simplifying the problem even further:

Given a set B with elements from [0, 6], group them into as many groups as possible such that the sum of each group is 0 (modulo 7).

B: 1 5 4 4



Simplifying the problem even further:

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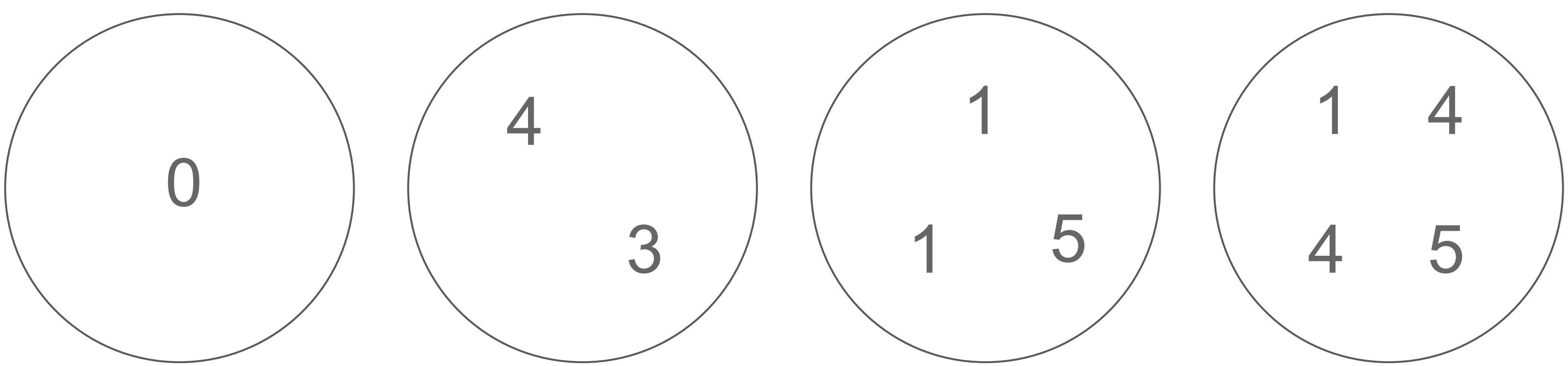
The solution is then N − number of groups = 10 - 4 = 6



To find the optimal grouping of numbers we start greedy:

1) As long as we have a zero in the set, make a group with a single zero in it.

2) As long as there is a pair of numbers that add up to 7 (1 and 6, 2 and 5, 3 and 4), make a group with these two numbers in it.

At this point the numbers in our set come from a set of at most three distinct integers: no zeros, either ones or sixes, either twos or fives, either threes or fours.

There exists a greedy O(N) strategy we could follow, but it's rather hard to find. Instead we may use a O(N^3) dynamic programming to complete the assignment.